Test Corrections

1. On any questions where points are missed, on a separate piece of paper, redo the question completely.

2. Identify (in sentences) the mistakes made on the test.

If both of these are correct for a question, I will give up to $\frac{1}{2}$ of the points back.

3. You can attempt this on each question as many times as needed by March 17^{th}. I would be glad to help.

Each time, turn in the original test + any corrections.

Back to interpolation:

Last time we found this formula for the equation of a line connecting two points (x_1, y_1), (x_2, y_2)

$$\Rightarrow \boxed{\begin{array}{l} y = \dfrac{y_1(x-x_2)}{(x_1-x_2)} + \dfrac{y_2(x-x_1)}{(x_2-x_1)} \\ \shortparallel \\ f(x) \end{array}}$$

Notice $f(x_1) = \dfrac{y_1 \cancel{(x_1-x_2)}}{\cancel{(x_1-x_2)}} + \dfrac{y_2 \overbrace{(x_1-x_1)}^{0}}{(x_2-x_1)}$

$= y_1$

$$f(x_2) = \frac{y_1 \overbrace{(x_2-x_2)}^{0}}{(x_1-x_2)} + \frac{y_2 \cancel{(x_2-x_1)}}{\cancel{(x_2-x_1)}}$$

$$= y_2$$

Generalization: what if I have 3 or more points.

- 3 points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

Find a polynomial of the form

$$f(x) = Ax^2 + Bx + C$$

that goes through all 3 points.

i.e. we want

$$y_1 = Ax_1^2 + Bx_1 + C$$
$$y_2 = Ax_2^2 + Bx_2 + C$$
$$y_3 = Ax_3^2 + Bx_3 + C$$

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3 unknowns A, B, C
3 equations.

(in fact) $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Note: The determinant of this matrix is $\neq 0$ iff x_1, x_2, x_3 are different.

Lagrange interpolation method to find the polynomial $f(x)$ that goes through the 3 points:

$$f(x) = y_1 \cdot \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$f(x) = \sum_{j=1}^{3} y_j \prod_{k \neq j} \frac{(x-x_k)}{(x_j-x_k)}$$

By construction, $f(x)$ is a 2^{nd} degree polynomial with $f(x_1) = y_1(1) + 0 + 0$ ✓

$f(x_2) = y_2$ ✓

$f(x_3) = y_3$ ✓.

General formula: Find polynomial $f(x)$ of degree $N-1$ that goes through $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$

<u>Ans:</u> Lagrange interpolation formula.

$$f(x) = \sum_{j=1}^{N} y_j \underbrace{\prod_{k \neq j} \frac{(x - x_k)}{(x_j - x_k)}}_{\text{Note } (N-1)\text{-degree.}}$$

<u>Example</u> Find a quadratic polynomial that goes through $(-1, 7), (2, 1), (3, 1)$.

$$f(x) = \sum_{j=1}^{N} y_j \prod_{k \neq j} \frac{(x - x_k)}{(x_j - x_k)}$$

$$= y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$(-1,7),\ (2,1),\ (3,1)$

$$f(x) = 7\,\frac{(x-2)(x-3)}{(\underset{-3}{-1-2})(\underset{-4}{-1-3})} + 1\,\frac{(x+1)(x-3)}{(\underset{3}{2+1})(\underset{-1}{2-3})}$$

$$+\ 1\,\frac{(x+1)(x-2)}{(\underset{4}{3+1})(\underset{1}{3-2})}$$

$$f(x) = \frac{7}{12}(x-2)(x-3) + -\frac{1}{3}(x+1)(x-3)$$

$$+\ \frac{1}{4}(x+1)(x-2)$$

$$f(x) = \frac{7}{12}\left(x^2-5x+6\right) + -\frac{1}{3}\left(x^2-2x-3\right)$$

$$+\ \frac{1}{4}\left(x^2-x-2\right)$$

$$\boxed{f(x) = \frac{1}{2}x^2 - \frac{5}{2}x + 4}$$

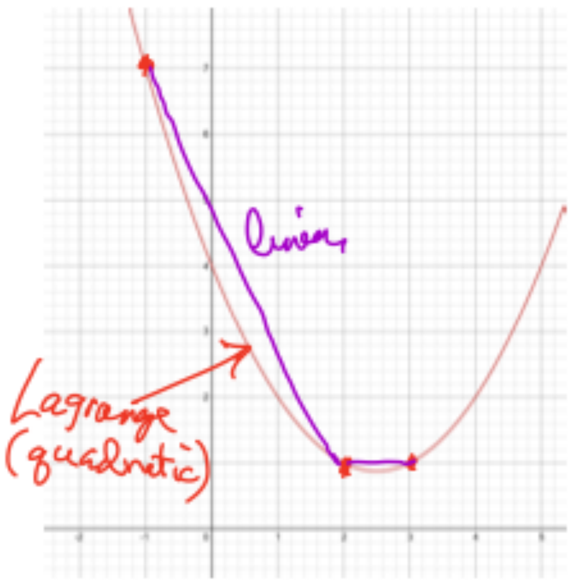
$$\begin{array}{r} -35 \\ +8 \\ -3 \\ \hline -30 \end{array}$$

$$\begin{array}{r} 42 \\ 12 \\ -6 \\ \hline 48 \end{array} \qquad \frac{-30}{12} = \frac{-10}{4}$$

$f(-1) = \frac{1}{2} + \frac{5}{2} + 4 = 7.$ ✓

$f(2) = 2 - 5 + 4 = 1.$ ✓

$f(3) = \frac{9}{2} - \frac{15}{2} + 4 = 1.$ ✓



This is our $f(x)$ that interpolates between the data points $(-1,7)$, $(2,1)$, $(3,1)$.

$f(x)$ as computed gives a way to approximate the data at other values of x.

Example: Using this function,

if $x_4 = 1$, $y_4 \approx f(x_4) = f(1)$

$= \frac{1}{2} - \frac{5}{2} + 4 = 2$.

if $x_5 = 0$, $y_5 \approx f(x_5) = 4$.

if $x_6 = 10$, $y_6 \approx f(x_6) = \frac{1}{2}(10)^2 - \frac{5}{2}(10) + 4$

$= 50 - 25 + 4 = 29$.

called an extrapolation — beyond our given data.

Question: How do these answers compare to linear interpolation between each pair of points?

Between $(-1,7)$ and $(2,1)$, the linear interpolation is

$$g(x) = y_1 \frac{(x-x_2)}{(x_1-x_2)} + y_2 \frac{(x-x_1)}{(x_2-x_1)}$$

$$= 7\frac{(x-2)}{\underbrace{(-1-2)}_{-3}} + 1\frac{(x+1)}{\underbrace{(2+1)}_{3}}$$

$$g(x) = -\frac{7}{3}(x-2) + \frac{1}{3}(x+1)$$

$$\boxed{g(x) = -2x+5}$$

$$\frac{14}{3}+\frac{1}{3}$$

$g(x_4) = -2(1)+5 = 3$ ← $f(x_4)$ was 2

$g(x_5) = -2(0)+5 = 5$ ← $f(x_5)$ was 4

Question: How can we quantify the error in a polynomial interpolation?

Newton's Finite Difference Formula

for polynomial interpolation:

This formulation will only work if the x-values are equally spaced

(eg $x_1 = 2$, $x_2 = 2.5$, $x_3 = 3$, $x_4 = 3.5$, etc)

ie. $x_j - x_{j-1} = \underset{\Delta x}{h} = \text{constant}$

Data $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$

$$\Delta x = h = x_1 - x_0 = x_2 - x_1, = x_3 - x_2, \ldots$$

x_j	y_j	Δy_j	$\Delta^2 y_j$	$\Delta^3 y_j$
x_0	y_0	$y_1 - y_0$	$(y_2 - y_1) - (y_1 - y_0)$ $y_2 - 2y_1 + y_0$	$(y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$ $y_3 - 3y_2 + 3y_1 - y_0$
x_1	y_1	$y_2 - y_1$	$(y_3 - y_2) - (y_2 - y_1)$ $y_3 - 2y_2 + y_1$	$y_4 - 3y_3 + 3y_2 - y_1$.
x_2	y_2	$y_3 - y_2$	$y_4 - 2y_3 + y_2$	$y_5 - 3y_4 + 3y_3 - y_2$
x_3	y_3	$y_4 - y_3$	$y_5 - 2y_4 + y_3$	
x_4	y_4	$y_5 - y_4$		
x_5	y_5			

Newton's interpolation formula using finite differences:

$$f(x) = \underset{f(x_0)}{y_0} + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$

$$+ \frac{1}{3!}\frac{\Delta^3(y_0)}{h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots + \frac{1}{N!}\frac{\Delta^N(y_0)}{h^N}\cdot(x-x_0)\cdot(x-x_1)\cdots(x-x_{N-1})$$

poly. of deg N that goes thru $(x_0, y_0), \ldots (x_{N-1}, y_{N-1}), (x_N, y_N)$

$N+1$ pts.

Great news: If $F(x)$ is the true function we are approximating, then

$(N+1)^{st}$ derivative of real fcn F

$$F(x) - f(x) = \frac{F^{(N+1)}(c)}{(N+1)!}(x-x_0)\cdots(x-x_N)$$

where c is somewhere between x_0 & x_N.

Remainder formula.